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Rational Finite Difference Approximation of High Order Accuracy for Nonlinear Two Point Boundary Value Problems

(Penghampiran Beza Terhingga Rasional Ketepatan Peringkat Tinggi untuk Masalah Nilai Dua Titik Sempadan Tak Linear)

P.K. Pandey*

ABSTRACT

In this paper, we present a new method for solving nonlinear general two point boundary value problems. A method based on finite differences and rational function approximation and we call this method as rational approximation method. A rational approximation method is applied to construct the numerical solution for two point boundary value problems. The novel method is tested on three model problems. Thus the numerical results obtained for these model problems show the performance and efficiency of the developed method.

Keywords: Boundary value problems; fourth-order method; rational approximation

ABSTRAK

Dalam kertas ini, kami memberikan kaedah baru bagi menyelesaikan masalah nilai dua titik sempadan tak linear umum. Kaedah berdasarkan beza terhingga dan penghampiran fungsi rasional ini dikenali sebagai kaedah penghampiran rasional. Kaedah penghampiran rasional yang digunakan untuk membina penyelesaian berangka bagi masalah dua titik sempadan. Kaedah baru ini diuji pada tiga model masalah. Oleh itu keputusan berangka yang diperoleh bagi masalah model ini menunjukkan prestasi dan keberkesanan kaedah yang dibangunkan.

Kata kunci: Kaedah peringkat empat; masalah nilai sempadan; penghampiran rasional

INTRODUCTION

The two point boundary value problems occur in all branches of sciences and technology. The majority of such problems are nonlinear and complicated. In general, analytical solution of these problems does not exist. So it is essential to obtain a solution by numerical methods. Consider a second order boundary value problem of the form:

$$y'' = f(x, y, y'), \qquad a \le x \le b, \tag{1}$$

subject to boundary conditions

$$y(a) = \alpha \text{ and } y(b) = \beta, \tag{2}$$

are real finite constants, $y(x), f(x, y, y') \in [a, b]$.

The existence and uniqueness of the solution, for the problem (1) is assumed. The specific theorems which list the conditions for existence and uniqueness of the solution of such problems are discussed in Baxley (1981), Keller (1968) and Stoer and Bulirsch (1991). The literature of numerical analysis contains much on the solution of the second order boundary value problems in Collatz (1966), Jain (1984), Keller (1968) and Lambert (1991). In this article we emphasized our self on iterative method only.

The iterative methods for the numerical solution of the problem (1) are proposed in Chawla (1978) and Tirmizi and Twizel (2002). However these methods are based on polynomial functions, which are normally smooth and with sufficient continuous derivatives. Thus these methods are exact for some degree of the polynomials. On the other hand a new approach reported in literature for first order initial value problems and based on non-polynomial functions (Okosun & Ademiluyi 2007; Ying & Yaacob 2013) and references there in.

In this paper, a new class of novel nonstandard rational finite difference method (Pandey 2013), the advancement of idea based on non-polynomial approximation (Mickens 1994; Ramos 2007) will be used for solving nonlinear two point boundary value problems. Newton-Raphson method is considered as a procedure for solving the nonlinear equation. It provides a promising tool in iterative solution field and open up almost new concept in numerical method.

The next section gives ways for constructing the proposed method. In the section that follows we discuss the local truncation error and in the subsequent section, the numerical results obtained for model problems. In final section some discussions and conclusion.

RATIONAL APPROXIMATION METHOD

Let set a uniform mesh $x_i = ih$, $i = 0, 1, \dots \dots, N-1, N$, and $Nh = (b-a)$ in [a, b].

Let's denote with y_i, an approximation of $y(x)$, $y_i' = y'(x_i)$, $f_i = f(x_i, y_i, y_i')$.

In this section, we discuss the derivation of the fourth order rational difference method for the problem (1) at the point $x = x_i$, when f is function of x and y only i.e. $f(x, y)$ as in Ying and Yaacob (2013), then rational difference method is,

$$y_{i+1} - 2y_i + y_{i+1} = \frac{12h^2 (f_i)^2}{14f_i - f_{i+1} - f_{i-1}}. \tag{3}$$

In order to discretize problem (1) at $= x_i$, we use the following $O(h^2)$ approximations as used by Chawla (1978),

$$\bar{y}'_i = \frac{y_{i+1} - y_{i-1}}{2h}. \tag{4.1}$$

$$\bar{y}'_{i+1} = \frac{3y_{i+1} - 4y_i + y_{i-1}}{2h}. \tag{4.2}$$

$$\bar{y}'_{i-1} = \frac{-y_{i+1} + 4y_i - 3y_{i-1}}{2h}. \tag{4.3}$$

So if we define and set a forcing function as in Chawla (1978),

$$\bar{f}_i = f\left(x_i, y_i, \bar{y}'_i\right). \tag{5.1}$$

$$\bar{f}_{i+1} = f\left(x_{i+1}, y_{i+1}, \bar{y}'_{i+1}\right). \tag{5.2}$$

$$\bar{f}_{i-1} = f\left(x_{i-1}, y_{i-1}, \bar{y}'_{i-1}\right). \tag{5.3}$$

It follows from (4.1)-(4.3), that $\bar{f}_1, \bar{f}_{i+1}, \bar{f}_{i-1}$ will provide $O(h^2)$ approximation for f_i, f_{i+1}, f_{i-1}, respectively.

Define

$$\bar{\bar{y}}'_i = \bar{y}'_i + a\,h\left(\bar{f}_{i+1} - \bar{f}_{i-1}\right), \tag{6}$$

where 'a' is free parameter to be determined.

We need $O(h^4)$ approximation for function $f(x_i, y_i, y_i')$, so set and define a forcing function as,

$$\bar{\bar{f}}_i = f\left(x_i, y_i, \bar{\bar{y}}'_i\right). \tag{7}$$

It follows from (5) and (6), that

$$\bar{\bar{f}}_i = f_i + h^2\left(\frac{1}{6} + 2a\right) y'''_i \frac{\partial f}{\partial y'_i} + O\left(h^4\right). \tag{8}$$

It follows from (8) that will provide an for if

$$12a + 1 = 0. \tag{9}$$

Thus the given problem (1), at central point $x = x_i$ may be discretized as:

$$y_{i+1} - 2y_i + y_i - 1 = \frac{12h^2\left(\bar{\bar{f}}_i\right)^2}{14\bar{\bar{f}}_i - \bar{f}_{i+1} - \bar{f}_{i-1}}. \tag{10}$$

Thus from (9), we conclude that order of proposed method (10) is four.

LOCAL TRUNCATION ERROR

The local truncation error of the proposed difference method (10), as in Pandey, (2013) and Jain et al. (1987) may be written as:

$$\begin{aligned} T_i &= y_{i+1} - 2y_i + y_{i-1} - \frac{12h^2\left(\bar{\bar{f}}_i\right)^2}{14\bar{\bar{f}}_i - \bar{f}_{i+1} - \bar{f}_{i-1}} \\ &= y_{i+1} - 2y_i + y_{i-1} - \frac{12h^2\left(\bar{\bar{f}}_i\right)^2}{12\bar{\bar{f}}_i - \left(\bar{f}_{i+1} - 2\bar{\bar{f}}_i + \bar{f}_{i-1}\right)}. \end{aligned} \tag{11}$$

Let us define $h^2 \bar{\bar{f}}''_i = \left(\bar{f}_{i+1} - 2\bar{\bar{f}}_i + \bar{f}_{i-1}\right)$ and substitute in (11), we will obtain:

$$T_i = y_{i+1} - 2y_i + y_{i-1} - \frac{12h^2\left(\bar{\bar{f}}_i\right)^2}{12\bar{\bar{f}}_i - h^2\bar{\bar{f}}''_i}.$$

Thus from (8-9) we have

$$\begin{aligned} T_i &= y_{i+1} - 2y_i + y_{i-1} - \frac{12h^2\left(f_i\right)^2}{12f_i - h^2 f''_i} \\ &= \mathrm{y}_{i+1} - 2\mathrm{y}_i + \mathrm{y}_{i-1} - \frac{12.h^2\left(\mathrm{y}''_i\right)^2}{12\mathrm{y}''_i - h^2.\mathrm{y}^{(4)}_i}. \end{aligned} \tag{12}$$

Expanding each term on the right side of (12) in Taylor series about point y_i, so we have,

$$\begin{aligned} \mathrm{T}_i &= \left(h^2\mathrm{y}''_i + \frac{h^4}{12}\mathrm{y}^{(4)}_i + \frac{h^6}{360}\mathrm{y}^{(6)}_i + \mathrm{o}\left(h^8\right)\right) \\ &\quad - h^2\mathrm{y}''_i\left(1 - \frac{h^2}{12}\frac{\mathrm{y}^{(4)}_i}{\mathrm{y}''_i}\right)^{-1}. \end{aligned} \tag{13}$$

Assuming that $\left|1 - \frac{h^2}{12}\frac{\mathrm{y}^{(4)}_i}{\mathrm{y}''_i}\right| < 1$, then by application of binomial theorem on right side of (18) and simplify, we will get,

$$\begin{aligned} T_i &= \frac{h^6}{360}\mathrm{y}^{(6)}_i - \frac{h^6}{144}\frac{\left(\mathrm{y}^{(4)}_i\right)^2}{\mathrm{y}''_i} + o\left(h^8\right). \\ T_i &= \frac{h^6}{72}\left(\frac{1}{5}\mathrm{y}^{(6)}_i - \frac{1}{2}\frac{\left(\mathrm{y}^{(4)}_i\right)^2}{\mathrm{y}''_i}\right) + o\left(h^8\right). \end{aligned} \tag{14}$$

Thus from (14), the order of the method (10) is four.

NUMERICAL RESULTS

In this section, the proposed method (10) is applied to solve three different model problems. All computations were carried out using double precision GNU FORTRAN language. Also Chawla's method (1978) is applied to solve these model problems. Let y_i an approximate value of the theoretical solution $y(x)$ at the point $x = x_i$. Maximum/ Minimum absolute errors

$$MAE(y) = \max_{1 \le i \le N-1} |y(x_i) - y_i|.$$
$$MIE(y) = \min_{1 \le i \le N-1} |y(x_i) - y_i|,$$

are shown for different step lengths h, in Tables 1-3 .

Example 1

Consider the equation

$$y'' = y^3 - yy', \quad 1 \le x \le 2,$$

subject to the boundary condition $y(1) = \frac{1}{2}, y(2) = \frac{1}{3}$. The exact solution of the problem is $y(x) = \frac{1}{1+x}$. In Table 1, maximum / minimum absolute errors of the present method (10) and Chawla's method presented.

Example 2

Consider the equation

$$y''(x) = \frac{3}{y}(y')^2, \quad 0 \le x \le 1,$$

subject to the boundary condition $y(0) = 1, y(1) = \frac{1}{\sqrt{2}}\cdot$ Let solution of the problem is $y(x) = \frac{1}{\sqrt{1+x}}\cdot$ In Table 2, maximum / minimum absolute errors of the present method (10) and Chawla's method presented.

Example 3

Consider the equation

$$y''(x) = -\frac{x}{\sqrt{1-y}}y' + f(x), \; 0 \le x \le 1,$$

subject to the boundary condition $y(0) = 0, y(1) = -3$. Let solution of the problem is $y(x) = 1 - (x^2 + 1)^2$. In Table 3, maximum/minimum absolute errors of the present method (10) and Chawla's method presented.

Conclusion

In this article, a non-classical rational method of order four for numerical solution of general two point boundary value problems was described. In general each numerical method has its own merit and demerit in its application. The present method was based on non- polynomial approximation. In the development of the method extra assumption used than

TABLE 1. Maximum absolute error $y(x) = \frac{1}{1+x}$ in for example 1

Errors		N			
		4	8	16	32
(10)	MAE	.25099516(-4)	.56059739(-5)	.62916013(-6)	.32939408(-7)
	MIE	.15686859(-4)	.18542228(-6)	.10842972(-6)	.32045508(-9)
Chawla's	MAE	.26351214(-4)	.56655786(-5)	.62916013(-6)	.32939408(-7)
	MIE	.16431917(-4)	.18840251(-5)	.10842972(-6)	.32045508(-9)

TABLE 2. Maximum absolute error in $y(x) = \frac{1}{\sqrt{1+x}}$ for example 2

Errors		N			
		8	16	32	64
(10)	MAE	.12934449(-3)	.61587798(-5)	.25958715(-6)	.29380638(-7)
	MIE	.27492788(-4)	.31613180(-5)	.16825087(-8)	.16825087(-8)
Chawla's	MAE	.21770765(-6)	.74650984(-7)	.14037788(-6)	.21254630(-6)
	MIE	.44558302(-7)	.17713894(-9)	.18189110(-9)	.16825087(-8)

TABLE 3. Maximum absolute error in for example 3

Errors		N			
		4	8	16	32
(10)	MAE	.23446083(-2)	.45490265(-3)	.53524971(-4)	.47683716(-5)
	MIE	.14810413(-2)	.10676309(-3)	.22007152(-5)	.35623088(-7)
Chawla's	MAE	.74594378(0)	.14065456(0)	.14402390(-1)	.10793209(-2)
	MIE	.43773103(0)	.19868940(-1)	.12140172(-2)	.76146564(-4)

that of finite difference method which may be considered as demerit of the method. Order of the method was estimated from numerical results and local truncation error. Method is computationally efficient, accurate and effective which can be observed in numerical results obtained in our experiments. It was observed from the results presented in table that method has higher accuracy i.e. small discretization error which may be considered as merit of the method. Thus we can say that the present method (10) seems competitive with other finite difference method. The range of absolute errors can be clearly seen from these tables. It is observed from computational results that max/min errors occur near boundary points. Our future works will deal with how to improve the accuracy of the method and to minimize the variation in max and min errors. The method may therefore be found useful in the solution of the problems, the result required to be accurate in term of minimum error. May it possible in future, a better results can be obtained using computing software other than GNU FORTRAN.

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Department of Mathematics
College of Applied Sciences
PB No. 1905, PC 211, Salalah
Sultanate of Oman

Department of Mathematics
Dyal Singh College (Univ. of Delhi)
Lodhi Road
New Delhi - 110003
India

*Corresponding author; email: pramod_10p@hotmail.com

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